# Optimal Mechanisms for Robust Coordination in Congestion Games

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***Abstract*—Uninfluenced social systems often exhibit suboptimal performance; specially-designed taxes can influence agent choices and thereby bring aggregate social behavior closer to optimal. A perfect system characterization may enable a planner to apply simple taxes to incentivize desirable behavior, but system uncertainties may necessitate highly-sophisticated taxation methodologies. Using a model of network routing, we study the effect of system uncertainty on a designer's ability to influence behavior with financial incentives. We show that in principle, it is possible to design taxes that guarantee that selfish network flows are arbitrarily close to optimal flows, despite the fact that agents' tax-sensitivities and network topology are unknown to the designer. In general, these taxes may be large; accordingly, for affine-cost parallel-network routing games, we explicitly derive the optimal bounded tolls and the best-possible performance guarantee as a function of a toll upper-bound. Finally, we restrict attention to simple fixed tolls and show that they fail to provide strong performance guarantees if the designer lacks accurate information about network topology or user sensitivities.**

## I. Introduction

It is well-known that in systems that are driven by social behavior, lack of coordination and agents' self-interested behavior can significantly degrade system performance. This poor performance is commonly referred to as the *price of anarchy*, defined as the ratio between the worst-case social welfare resulting from selfish behavior and the optimal social welfare [2]. This degradation of performance due to selfish behavior has been the subject of research in areas of network resource allocation [3], distributed control [4], traffic congestion [5], [6], and others. As a result, there is a growing body of research geared at influencing social behavior to improve system performance [7]–[13].

To study the issues surrounding the problem of influencing selfish social behavior, we turn to a simple model of traffic routing: a mass of traffic needs to be routed across a network in a way that minimizes the average network transit time. If a central planner can direct traffic explicitly, it is straightforward to compute the routing profile that minimizes total congestion. However, in real systems, it may not be possible to implement such direct centralized control or prescribe such optimal coordinated behavior: for example, if the network represents a city's road network, individual drivers make their own routing choices in response to their own personal objectives.

Accordingly, we may model this routing problem as a nonatomic congestion game, where the traffic can be viewed as a collection of infinitely-many users, each controlling an infinitesimally-small amount of traffic and seeking to minimize its own transit time. We use the popular concept of a *Nash flow* (defined as a routing profile in which no user can switch to a different path and decrease her transit delay) to characterize the routing profile resulting from such self-interested behavior. It is widely known that Nash flows can exhibit considerably higher congestion than optimal flows. An important result in this setting states that a Nash flow on a network with general latency functions can be arbitrarily worse than an optimal flow [14]. That is, the price of anarchy is unbounded; this is true even on networks consisting of only two links. Recent research has investigated the price of anarchy of transportation networks under various conditions [15]–[18].

A separate research agenda has investigated methods of incentivizing individual network users to choose more-efficient routes, thereby aligning Nash flows with optimal flows. This can be viewed as an attempt to incentivize coordination between the users of the network. A natural approach to this is to charge monetary taxes for the use of network links. Existing research has explored methods of designing such optimal taxes given that the tax-designer has access to certain information regarding the system. In [19]–[21] it is shown that optimal "fixed" taxes (i.e., taxes are constant functions of traffic flow) can be computed for any routing game, but the computation requires precise characterizations of the network topology, user demands, and user tax-sensitivities. In contrast, [22], [23] derive optimal taxes known as "marginal-cost taxes" which require no knowledge of the network topology or user demands, but require that all users share a common known tax-sensitivity. Furthermore, the marginal-cost taxation functions must be strictly flow-varying. Section III details these results.

In this paper, we ask if it is possible to compute optimal taxes with minimal information about the system, and present several new results showcasing the relationship between available tolling methodologies, uncertainty, and achievable performance. We term this goal "robust coordination," as we desire to incentivize agents to behave as though they are coordinating with one another, but we require that our behavior-influencing mechanisms are robust to mischaracterizations of the system. Since price of anarchy is simply a cost metric in worst-case over some set of unknown information, it lends itself naturally to quantifying the robustness of taxation mechanisms to unknown information. Thus, our analysis represents a departure

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both from the typical descriptive price of anarchy research as well as from the complete-information assumptions of the taxation literature.

Our main contribution is to derive a universal taxation mechanism that guarantees arbitrarily-good performance for any routing game while requiring no prior knowledge of the specific network, user demand profile, or distribution of user sensitivities. That is, our derived taxes are robust to gross mischaracterizations of the above quantities. This result holds for networks with general latency functions and any topology, suggesting that surprisingly-little information is required in principle.

Our next result explores the effect of reducing the designer's capabilities while maintaining a high level of uncertainty. To this end, our second contribution is to explore the effect of placing an upper bound on the allowable tolling functions. This may have practical value in settings where very large tolls may be impossible (or politically unpalatable) to implement. For parallel networks with linear-affine latency functions, we derive the optimal tolling functions that minimize worst-case performance degradation for any unknown distribution of user sensitivities and toll upper bound, requiring no prior knowledge of the number of network links. These optimal tolls are simple affine functions of flow. We show that for parallel networks with linear-affine cost functions and simple user demands, the worst-case performance degradation strictly decreases with the toll upper bound. Our results suggest that large tolls can compensate for a poor characterization of user sensitivities. Unfortunately, by imposing an upper bound on allowable taxation functions, optimal behavior can no longer be guaranteed. Thus, this result additionally implies that unbounded tolls are *necessary* to enforce optimal flows if both the network topology and user sensitivities are unknown.

Our results in Section VI explore a further restriction on the designer's capabilities, requiring that tolls do not depend on flow (i.e., requiring fixed tolls rather than tolling functions). These results suggest that fixed tolls lack robustness to mischaracterizations of network topology and user sensitivity. First, if the network topology is unknown, fixed tolls *cannot* enforce perfectly optimal routing, and we present a simple setting in which network performance can be arbitrarily bad if fixed tolls are not allowed to depend on the network structure. Finally, we show that even if fixed tolls are allowed to depend on the network topology and user demands, they provide relatively poor performance guarantees when the user sensitivities are unknown. Here, by reducing the designer's capability (by disallowing access to flow-varying taxation functions), we dramatically reduce the achievable performance guarantees in the presence of uncertainty. That is, fixed tolls are significantly less robust than flow-varying tolls. Our negative result here vividly demonstrates the need for a clear understanding of the robustness of incentive mechanisms to model imperfections.

## II. Model and Performance Metrics

### A. Routing Game

Consider a network routing problem in which a unit mass of traffic needs to be routed across a network $(V, E)$, which consists of a vertex set $V$ and edge set $E \subseteq (V \times V)$. We call a source/destination vertex pair $(s^c, t^c) \in (V \times V)$ a *commodity*, and the set of all such commodities $\mathcal{C}$. For each $c \in \mathcal{C}$, there is a mass of traffic $r^c > 0$ that needs to be routed from $s^c$ to $t^c$. We write $\mathcal{P}^c \subset 2^E$ to denote the set of *paths* available to traffic in commodity $c$, where each path $p \in \mathcal{P}^c$ consists of a set of edges connecting $s^c$ to $t^c$. Let $\mathcal{P} = \cup\{\mathcal{P}^c\}$.

We write $f_p^c \geq 0$ to denote the mass of traffic from commodity $c$ using path $p$, and $f_p \triangleq \sum_{c\in\mathcal{C}} f_p^c$. A *feasible flow* $f \in \mathbb{R}^{|\mathcal{P}|}$ is an assignment of traffic to various paths such that for each $c$, $\sum_{p\in\mathcal{P}^c} f_p^c = r^c$. Without loss of generality, we assume that $\sum_{c\in\mathcal{C}} r^c = 1$.

Given a flow $f$, the flow on edge $e$ is given by $f_e = \sum_{p:e\in p} f_p$. To characterize transit delay as a function of traffic flow, each edge $e \in E$ is associated with a specific latency function $\ell_e : [0,1] \to [0,\infty)$. We adopt the standard assumptions that latency functions are nondecreasing, continuously differentiable, and convex. Note that latency functions are anonymous: all traffic affects delay equally. The cost of a flow $f$ is measured by the *total latency*, given by

$$\mathcal{L}(f) = \sum_{e\in E} f_e \cdot \ell_e(f_e) = \sum_{p\in\mathcal{P}} f_p \cdot \ell_p(f_p), \tag{1}$$

where $\ell_p(f) = \sum_{e\in p} \ell_e(f_e)$ denotes the latency on path $p$. We denote the flow that minimizes the total latency by

$$f^* \in \underset{f \text{ is feasible}}{\operatorname{argmin}} \ \mathcal{L}(f). \tag{2}$$

A *routing problem* is given by the tuple $G = (V, E, \mathcal{C}, \{\ell_e\})$. We write the set of all such routing problems as $\mathcal{G}$, and often write $e \in \mathcal{G}$ to denote $(e \in G : G \in \mathcal{G})$.

In this paper we study taxation mechanisms for influencing the emergent collective behavior resulting from self-interested price-sensitive users. To that end, we model the above routing problem as a non-atomic game in which the traffic models a large population of users. Thus, we use the terms "traffic," "users," and "agents" interchangeably. We assign each edge[1] $e \in E$ a flow-dependent, nondecreasing taxation function $\tau_e : [0,1] \to \mathbb{R}^+$. We characterize the taxation sensitivities of the users in commodity $c$ with a monotone, nondecreasing function $s^c : [0, r^c] \to [S_\mathrm{L}, S_\mathrm{U}]$, where each user $x \in [0, r^c]$ has a taxation sensitivity $s_x^c \in [S_\mathrm{L}, S_\mathrm{U}] \subseteq \mathbb{R}^+$ and $S_\mathrm{U} \geq S_\mathrm{L} \geq 0$ denote upper and lower sensitivity bounds, respectively. Given a flow $f$, the cost that user $x \in [0, r^c]$ experiences for using path $\tilde{p} \in \mathcal{P}^c$ is of the form

$$J_x^c(f) = \sum_{e\in\tilde{p}} \left[\ell_e(f_e) + s_x^c \tau_e(f_e)\right]. \tag{3}$$

Thus, for each user $x \in [0, r_c]$, the sensitivity $s_x^c$ can be viewed as a constant gain on the toll; a user's experienced cost is then the sum of the latency and sensitivity-weighted toll. Note that sensitivity can be interpreted as the reciprocal of an agent's value-of-time.[2] Note that $s_x^c$ need not equal $s_y^c$ for $x \neq y$. We

[1] Note that we allow all edges to be taxed (as in [19]–[23]); see [24] for a relaxation of this requirement.

[2] We adopt this formulation from [19]. Note that constant sensitivity is a commonly-studied special case; alternative formulations are possible [25].

assume that each user prefers the lowest-cost path from the available source-destination paths. We call a flow $f$ a *Nash flow* if for all commodities $c \in \mathcal{C}$ and all users $x \in [0, r^c]$ we have

$$J_x^c(f) = \min_{p \in \mathcal{P}^c} \left\{ \sum_{e \in p} \left[ \ell_e(f_e) + s_x^c \tau_e(f_e) \right] \right\}. \tag{4}$$

It is well-known that a Nash flow exists for any non-atomic game of the above form [26].

In our analysis, we assume that each sensitivity distribution function $s^c$ is unknown; for a given routing problem $G$ and $S_{\rm U} \geq S_{\rm L} \geq 0$ we define the set of possible sensitivity distributions as the set of monotone, nondecreasing functions $\mathcal{S}_G = \{s^c : [0, r^c] \to [S_{\rm L}, S_{\rm U}]\}_{c \in \mathcal{C}}$. We write $s \in \mathcal{S}_G$ to denote such a specific collection of sensitivity distributions, which we term a *population*.

### B. Price of Anarchy and Robustness

For a given routing problem $G \in \mathcal{G}$, we gauge the efficacy of a collection of taxation functions $\tau = \{\tau_e\}_{e \in E}$ by comparing the total latency of the resulting Nash flow and the total latency associated with the optimal flow, and then performing a worst-case analysis over all possible user populations. Let $\mathcal{L}^*(G)$ denote the total latency associated with the optimal flow, and $\mathcal{L}^{\rm nf}(G, s, \tau)$ denote the total latency of the worst-performing Nash flow resulting from taxation functions $\tau$ and population $s$. The worst-case system cost associated with this specific instance is captured by the *price of anarchy* which is of the form

$$\mathrm{PoA}(G, \tau) = \sup_{s \in \mathcal{S}_G} \left\{ \frac{\mathcal{L}^{\rm nf}(G, s, \tau)}{\mathcal{L}^*(G)} \right\} \geq 1. \tag{5}$$

In this context, we seek taxation mechanisms which minimize $\mathrm{PoA}(G, \tau)$ for a wide variety of routing games $G$. If a taxation mechanism $\tau$ brings $\mathrm{PoA}(G, \tau)$ close to 1 for many games $G$ (in a sense to be made exact later), this indicates that $\tau$ is robust to to mischaracterizations of user sensitivities. Traditionally, the price of anarchy is analyzed in worst case over a given class of games [14]. In our usage, we delay taking the worst case over all networks until specific settings when it is called for. For example, Theorem 1 exhibits a taxation mechanism which drives the price of anarchy to 1 for *every* $G$, but the rate at which the price of anarchy approaches 1 may vary from network to network. On the other hand, Theorem 3 provides an expression for the price of anarchy that holds for *all* parallel networks.

## III. Related Work

The following is a brief overview of the existing literature on taxation mechanisms in this context. A taxation mechanism simply computes edge tolls as a function of some set of information about the system; here, we focus in particular on the informational dependencies of several well-studied taxation approaches.

– *Omniscient taxation mechanisms:* These taxation mechanisms are assumed to have access to complete information regarding the routing game. For edge $e \in G$ and population $s \in \mathcal{S}_G$, the edge tolling function takes the following form: $\tau_e(f_e; G, s)$ . That is, each edge's taxation function can depend on the entire routing problem $G$ and the population sensitivities $s$. Recent results have identified taxation mechanisms of this form that assign fixed tolls (i.e., for any $e \in G$, $\tau_e(f_e) = q_e$ for some $q_e \geq 0$) that can enforce any feasible flow [20], [21], thus guaranteeing a price of anarchy of 1. However, the robustness of these mechanisms to variations or mischaracterizations of network topology and user sensitivities is heretofore unknown.

– *Network-agnostic taxation mechanisms:* This type of taxation mechanism is agnostic to network specifications: each taxation function is derived from locally-available information only. Here, a system designer essentially commits to a taxation function for each potential edge $e \in \mathcal{G}$, and any network realization $G \in \mathcal{G}$ merely employs a subset of these predefined taxation functions. An edge's toll cannot depend on any *other* edge's cost or location in the network, nor can it depend on the tax-sensitivities of the agents.

A commonly-studied network-agnostic taxation mechanisms is the marginal-cost (or Pigovian) taxation mechanism $\tau^{\rm mc}$, which is of the following form: for any $e \in \mathcal{G}$ with latency function $\ell_e$, the accompanying taxation function is

$$\tau_e^{\rm mc}(f_e) = f_e \cdot \frac{d}{df_e} \ell_e(f_e), \ \forall f_e \geq 0. \tag{6}$$

In [22] it is shown that for any $G \in \mathcal{G}$ we have $\mathcal{L}^*(G) = \mathcal{L}^{\rm nf}(G, s, \tau^{\rm mc})$ provided that all users have a sensitivity exactly equal to 1. Hence, irrespective of the underlying network structure, a marginal-cost taxation mechanism always ensures the optimality of the resulting Nash flow, provided that all users share a common known sensitivity.

There are many other results in this area; for example, in [27] the authors investigate the price of anarchy of various types of tolling functions with built-in upper bounds. In [28], it is shown that if taxes can be computed in a centralized fashion, any feasible flow can be enforced even if the central planner does not know the network's latency functions. For affine-cost parallel networks, [29] derives omniscient, flow-varying taxation mechanisms for applications where the total traffic rate is unknown. Finally, in [7], the authors show that marginal-cost taxes scaled by $\sqrt{S_{\rm L} S_{\rm U}}$ do possess a degree of robustness to mischaracterizations of user sensitivities for affine-cost parallel networks.

## IV. A Universal Taxation Mechanism

In this paper, we prove that network- and sensitivity-agnostic tolls exist which can drive the price of anarchy to 1 for general networks and latency functions. We term these "universal" because they take the same form and provide the same performance guarantee regardless of which particular routing scenario they are applied to. Using this taxation mechanism, we show in Theorem 1 that for any network, regardless of network topology, traffic rates $\{r^c\}$, or price-sensitivity functions $\{s^c\}$, the price of anarchy can be made arbitrarily close to 1 with sufficiently-large edge tolls, indicating that tolls exist which are robust to mischaracterizations of all the aforementioned system parameters.

**Theorem 1.** *Let $\mathcal{G}$ be the set of multi-commodity routing games where $S_{\mathrm{U}} \geq S_{\mathrm{L}} > 0$. For any network edge $e \in \mathcal{G}$ with convex, nondecreasing, continuously differentiable latency function $\ell_e$, define the universal taxation function on edge $e$ with gain parameter $\kappa \geq 0$ as*

$$\tau_e^{\mathrm{u}}(f_e;\kappa) = \kappa\left(\ell_e(f_e) + f_e \cdot \frac{d}{df_e}\ell_e(f_e)\right). \tag{7}$$

*Then for any routing problem $G \in \mathcal{G}$,*

$$\lim_{\kappa\to\infty} \mathrm{PoA}\left(G, \tau^{\mathrm{u}}(\kappa)\right) = 1. \tag{8}$$

That is, on *any* network being used by *any* population of users, the total latency can be made arbitrarily close to the optimal latency, and each individual link toll is a simple continuous function of that link's flow. The reason for this is that as $\kappa$ increases, the original latency function has a smaller and smaller relative effect on the users' cost functions; in the large-toll limit, the only cost experienced by the users is the tolling function itself which is specifically designed to induce optimal Nash flows.

*Proof.* Using a sequence of tolls, we construct a sequence of Nash flows that converges to an optimal flow. Let $\kappa_n$ be an unbounded, increasing sequence of tolling coefficients.

For any routing problem $G \in \mathcal{G}$ and price-sensitivities $s \in \mathcal{S}_G$, let $f^n = \left(f_p^n\right)_{p\in\mathcal{P}}$ denote the Nash flow resulting from the tolling coefficient $\kappa_n$. For each commodity $c$, let $\mathcal{P}_n^c \subseteq \mathcal{P}^c$ denote the set of paths that have positive flow in $f^n$. For any $p \in \mathcal{P}_n^c$, there must be some user $x \in [0, r^c]$ using $p$ with sensitivity $s_x^c$; the cost experienced by this user is given by

$$J_x^c(f^n) = \sum_{e\in p}\left[\ell_e(f_e) + \kappa_n s_x^c\left(\ell_e(f_e) + f_e \cdot \frac{d}{df_e}\ell_e(f_e)\right)\right].$$

Define $\gamma_{n,x} \triangleq \frac{\kappa_n s_x^c}{1+\kappa_n s_x^c}$. Let $\ell_e^*(f_e) = f_e \cdot \frac{d}{df_e}\ell_e(f_e)$; then for any other path $p' \in \mathcal{P}^c \setminus p$, user $x$ must experience a lower cost on $p$ than on $p'$, or

$$\sum_{e\in p}\ell_e(f_e) - \sum_{e\in p'}\ell_e(f_e) \leq \gamma_{n,x}\left[\sum_{e\in p'}\ell_e^*(f_e) - \sum_{e\in p}\ell_e^*(f_e)\right]. \tag{9}$$

Therefore, for any $n \geq 1$, $f^n$ must satisfy some set of inequalities defined by (9). Note that for all $c \in \mathcal{C}$ and any $x \in [0, r^c]$, $\lim_{\kappa_n\to\infty}\gamma_{n,x} = 1$, so because all the functions in (9) are continuous, $f^n$ converges to a set $F^*$ of feasible flows that satisfy

$$\sum_{e\in p}\ell_e(f_e) - \sum_{e\in p'}\ell_e(f_e) \leq \left[\sum_{e\in p'}\ell_e^*(f_e) - \sum_{e\in p}\ell_e^*(f_e)\right] \tag{10}$$

for all $c$, all $p \in \mathcal{P}^{c*}$, and $p' \in \mathcal{P}^c$, where $\mathcal{P}^{c*} \subseteq \mathcal{P}^c$ is some subset of paths. But inequalities (10) (combined with the feasibility constraints on $f$) also specify a Nash flow for $G$ for a unit-sensitivity population with marginal-cost taxes as specified by (6). Any such Nash flow must be optimal [22]; that is, any $f \in F^*$ is a minimum-latency flow for $G$. Thus, since $\mathcal{L}(f)$ is a continuous function of $f$,

$$\lim_{n\to\infty}\mathcal{L}\left(f^n\right) = \mathcal{L}^*\left(G\right), \tag{11}$$

obtaining the proof of the theorem. □

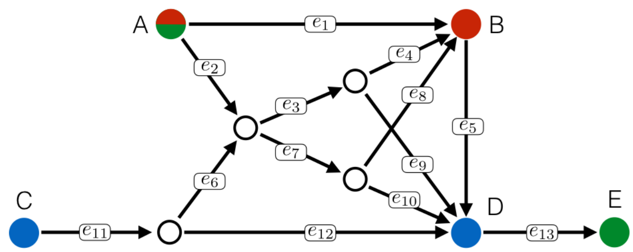


Fig. 1. Base network for Examples 1 and 2. This network has three commodities (i.e., source-destination pairs): $(A, B)$ (red), $(A, E)$ (green), and $(C, D)$ (blue), with associated traffic rates $r_1$, $r_2$, and $r_3$, respectively. Traffic in each commodity has access to all directed paths that connect the respective source and destination; for example, $(A, B)$ can choose between $\{e_1\}$, $\{e_2, e_3, e_4\}$, and $\{e_2, e_7, e_8\}$. For a demonstration of universal tolls applied to a specific instance of this network, see Figure 2. To demonstrate the effects of the universal tolls of Theorem 1, random variations of this network are simulated and the resulting price-of-anarchy values are plotted in Figure 3.

### A. Price of Anarchy Bounds for Homogeneous Populations

The result in Theorem 1 is encouraging since it ensures that no routing game or user population is so pathological that we cannot enforce optimal routing with sufficiently-high tolls, but it gives no indication of *how high* these tolls must be. In our next result in Proposition 2 (which follows from a result in [30]), we state that for homogeneous price-sensitive populations (i.e., all users have the same non-zero price sensitivity), the performance degradation is uniformly bounded in all games by a simple expression.

**Proposition 2.** *If all users have (unknown) homogeneous price-sensitivity $s \geq S_{\mathrm{L}} > 0$, the price of anarchy induced by $\tau^{\mathrm{u}}(\kappa)$ is given by*

$$\sup_{G\in\mathcal{G}} \mathrm{PoA}\left(G, \tau^{\mathrm{u}}(\kappa)\right) \leq \frac{1+\kappa S_{\mathrm{L}}}{\kappa S_{\mathrm{L}}}. \tag{12}$$

*Proof.* Immediate from Proposition 6.4 of [30]. □

### B. Examples Illustrating Universal Tolls

**Example 1.** Consider the network in Figure 1. This network has been used to demonstrate dynamic tolling mechanisms [13], and we use variations of it to illustrate the universal tolls of Theorem 1. This network has three commodities, labeled in Figure 1 as $(A, B)$ (red), $(A, E)$ (green), and $(C, D)$ (blue), with respective traffic rates $r_1$, $r_2$, and $r_3$. Traffic in each commodity has access to all paths connecting its source to its destination; for example, the $(A, B)$ commodity (shown in red in Figure 1) has access to three paths: $\{e_1\}$, $\{e_2, e_3, e_4\}$, and $\{e_2, e_7, e_8\}$.

First, consider an instance of this network in which $r_1 = r_2 = r_3 = 0.5$, all traffic in commodity 1 has sensitivity $s_1 = 100$, and all traffic in commodities 2 and 3 have sensitivity $s_2 = s_3 = 0.1$. Let the latency functions be given by $\ell_e(f_e) = a_e\left(f_e\right)^4 + b_e$, with coefficients $a_e$ and $b_e$ given in Table I. Quartic latency functions of this form are a stylized form of the well-known Bureau of Public Roads (BPR) latency functions, commonly used to model the congestion characteristics of physical roads [13], [31]. The optimal flow on this network (Figure 2(a)) has a total latency of approximately 1.49; the un-influenced Nash flow

TABLE I
LATENCY AND UNIVERSAL TOLLING FUNCTIONS FOR EXAMPLE 1

| edge | $\ell_e(f_e) = a_e(f_e)^4 + b_e$ | $\tau_e^{\mathrm{u}}(f_e) = \kappa\left(5a_e(f_e)^4 + b_e\right)$ |
|---|---|---|
| $e_1$ | $0.88(f_{e_1})^4 + 0.10$ | $\kappa\left(4.40(f_{e_1})^4 + 0.10\right)$ |
| $e_2$ | $0.59(f_{e_2})^4 + 0.91$ | $\kappa\left(2.95(f_{e_2})^4 + 0.91\right)$ |
| $e_3$ | $0.66(f_{e_3})^4 + 0.87$ | $\kappa\left(3.30(f_{e_3})^4 + 0.87\right)$ |
| $e_4$ | $0.24(f_{e_4})^4 + 0.88$ | $\kappa\left(1.20(f_{e_4})^4 + 0.88\right)$ |
| $e_5$ | $0.57(f_{e_5})^4 + 0.93$ | $\kappa\left(2.85(f_{e_5})^4 + 0.93\right)$ |
| $e_6$ | $0.62(f_{e_6})^4 + 0.12$ | $\kappa\left(3.10(f_{e_6})^4 + 0.12\right)$ |
| $e_7$ | $0.89(f_{e_7})^4 + 0.34$ | $\kappa\left(4.45(f_{e_7})^4 + 0.34\right)$ |
| $e_8$ | $0.93(f_{e_8})^4 + 0.93$ | $\kappa\left(4.65(f_{e_8})^4 + 0.93\right)$ |
| $e_9$ | $0.68(f_{e_9})^4 + 0.22$ | $\kappa\left(3.40(f_{e_9})^4 + 0.22\right)$ |
| $e_{10}$ | $0.31(f_{e_{10}})^4 + 0.72$ | $\kappa\left(1.55(f_{e_{10}})^4 + 0.72\right)$ |
| $e_{11}$ | $0.26(f_{e_{11}})^4 + 0.40$ | $\kappa\left(1.30(f_{e_{11}})^4 + 0.40\right)$ |
| $e_{12}$ | $0.54(f_{e_{12}})^4 + 0.45$ | $\kappa\left(2.70(f_{e_{12}})^4 + 0.45\right)$ |
| $e_{13}$ | $0.06(f_{e_{13}})^4 + 0.08$ | $\kappa\left(0.30(f_{e_{13}})^4 + 0.08\right)$ |

(Figure 2(b)) has a total latency of approximately 1.955, for an un-influenced price of anarchy of about 1.31. Applying universal tolls (7) to this network (tolling functions in Table I) results in an improvement in the total latency; Nash flows for $\kappa = 0.5$ and $\kappa = 10$ are depicted in Figure 2(c) and (d), respectively. Figure 2(e) plots the price of anarchy of this specific instance as a function of $\kappa$. Values of $\kappa$ as low as 0.1 reduce the price of anarchy from 1.31 to 1.19; when $\kappa \geq 5$, the price of anarchy is already below 1.05. For comparison, the worst-case price of anarchy (over all networks) for quartic latency functions is approximately 2.15 [14].

**Example 2.** The price of anarchy curve in Figure 2(e) is specific to one particular routing problem and user population; different networks and populations have different curves. To study the effect of universal tolls on more than this single instance, networks were generated by randomly deleting edges from the network in Figure 1 (while requiring that each commodity has a feasible S-D path and that at least one commodity has more than one such path). BPR latency functions were chosen of the form $\ell_e(f_e) = a_e\left(f_e\right)^4 + b_e$, where the $a_e$ and $b_e$ coefficients were chosen independently and uniformly at random from the interval $[0, 1]$. Each network was simulated with several homogeneous and heterogeneous user populations and several different traffic rates, for a total of 2,314 individual routing problems simulated. The ratio of Nash total latency to optimal total latency (instance-specific price of anarchy) was computed on each of these 2,314 routing problems in response to universal tolls (7) for $\kappa \in \{0, 0.5, 1, 2, 5, 10\}$.

The results of the simulations are plotted in Figure 3. Each dot corresponds to the price of anarchy of a routing problem instance simulated for a given value of $\kappa$. Also included are lines corresponding to the maximum, 99th, and 75th percentile of simulated PoA values for each $\kappa$. Note in particular the red dotted 75th-percentile line which indicates that when $\kappa = 0.5$, 75% of the simulations reported a price of anarchy below 1.01. This is not an exhaustive search over all price of anarchy values for the considered networks; there may well be specific choices of latency functions and populations which result in higher congestion than plotted here. Nonetheless,

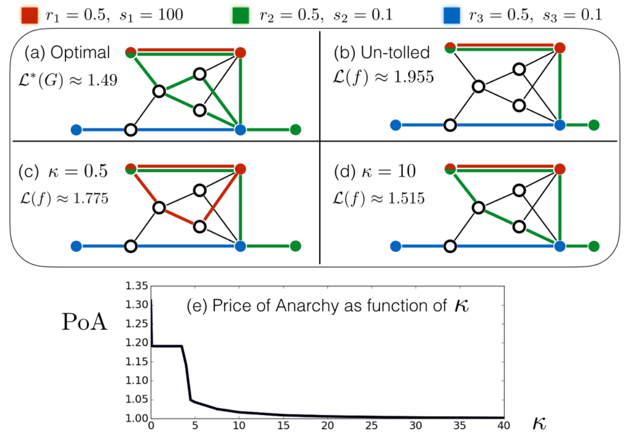


Fig. 2. Specific instance of network from Figure 1 for Example 1. Commodity traffic rates are $r_1 = r_2 = r_3 = 0.5$, and each commodity is assigned homogeneous sensitivity values of $s_1 = 100$ and $s_2 = s_3 = 0.1$. Latency functions of $\ell_e(f_e) = a_e\left(f_e\right)^4 + b_e$ were assigned to the network, with values as shown in Table I. The optimal flow on the network was computed, as well as Nash flows resulting from universal tolls (7) for several values of parameter $\kappa$. In (a)-(d), a colored edge indicates that traffic from the commodity corresponding to that color is using that edge; a thin black edge indicates no flow. Part (e) plots the simulated price of anarchy on this network, for this population, as a function of tolling parameter $\kappa$.

these simulations illustrate the concept that in many cases, relatively low values of $\kappa$ may suffice to incentivize nearly-optimal flows.

## V. THEOREM 3: OPTIMAL BOUNDED TOLLS

Of course, it may be impractical or politically infeasible to charge extremely high tolls. For example, if network demand is elastic, very large tolls could induce some users to avoid travel altogether. Therefore, in Theorem 3, we analyze the effect of an upper bound on the allowable tolling functions. For simplicity, we focus on parallel networks, which have been used to model problems such as scheduling small jobs on machines [32]. For parallel networks with affine cost functions in which every edge has positive flow in an un-tolled Nash flow, we explicitly derive the optimal bounded taxation mechanism, and then provide an expression for the price of anarchy. These optimal tolls are simple affine functions of flow, and the price of anarchy is strictly decreasing in the upper bound. Formally, we say a taxation mechanism is *bounded* if all its taxation functions respect some upper bound:

**Definition 1.** *Taxation mechanism $\tau$ is* bounded *by $T$ on a class of routing problems $\bar{\mathcal{G}}$ if for every edge $e \in \bar{\mathcal{G}}$, $\tau$ assigns a (possibly flow-varying) tolling function that satisfies*

$$\tau_e : [0, 1] \to [0, T]. \tag{13}$$

*$\mathcal{T}\left(T, \bar{\mathcal{G}}\right)$ denotes the set of mechanisms bounded by $T$ on $\bar{\mathcal{G}}$.*

For the following results, let $\mathcal{G}^p \subseteq \mathcal{G}$ represent the class of all single-commodity, parallel-link routing problems with affine latency functions. That is, for all $e \in \mathcal{G}^p$, the latency function satisfies

$$\ell_e(f_e) = a_e f_e + b_e \tag{14}$$

where $a_e \geq 0$ and $b_e \geq 0$ are edge-specific constants. "Single-commodity" implies that all traffic has access to all network

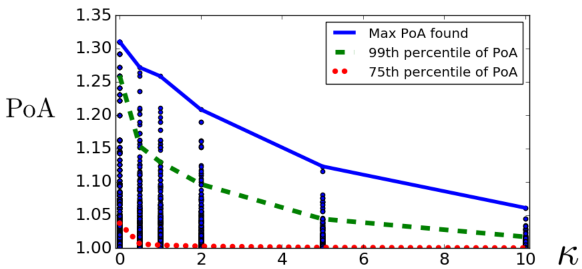


Fig. 3. Monte Carlo results for Example 2. Each blue dot represents the price of anarchy of one routing game in response to universal tolls (7) for a specific value of $\kappa$. The solid blue, dashed green, and dotted red lines represent respectively the maximum, 99th percentile, and 75th percentile of Price of Anarchy values found for each corresponding value of $\kappa$. Note in particular that the 75th percentile line is less than 1.01 for all $\kappa \geq 0.5$, suggesting that relatively low tolls may suffice for many networks.

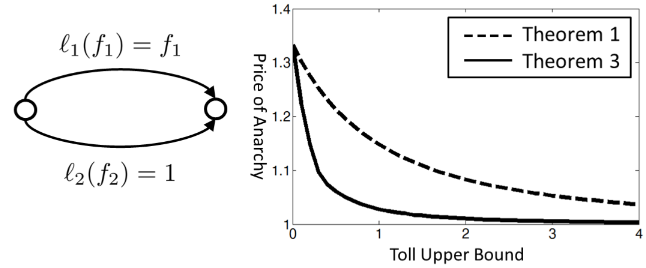


Fig. 4. Price of Anarchy plot contrasting the Universal toll result from Theorem 1 (dashed line) with the optimal toll result from Theorem 3 (solid line) on the special case of the two-link network depicted on the left. For both price of anarchy curves, the user sensitivities satisfy $S_{\rm L} = 1$ and $S_{\rm U} = 10$. The price of anarchy of either taxation mechanism converges to 1 as the toll upper bound increases, but the solid line converges much more quickly. This is because Theorem 3 gives the optimal tolls for a specific class of networks (parallel networks), but the universal tolls from Theorem 1 are designed to work on *all* classes of networks.

edges. Furthermore, we assume that every edge has positive flow in an un-tolled Nash flow.[3] In order to meaningfully discuss uniform toll bounds on a broad class of networks, it is necessary to describe classes of networks with bounded latency functions. To this end, we define $\mathcal{G}\left(\bar{a}, \bar{b}\right) \subset \mathcal{G}^p$ as the set of parallel, affine-cost networks such that for every $e \in \mathcal{G}\left(\bar{a}, \bar{b}\right)$, the latency function coefficients satisfy $a_e \leq \bar{a}$ and $b_e \leq \bar{b}$. Note that $\bar{a}$ and $\bar{b}$ represent the maximum-possible congestibility and free-flow time, respectively; estimates of these quantities should be available because they are functions of physical parameters such as distance and road width.

**Definition 2.** *For every edge $e \in \mathcal{G}$ with latency function $\ell_e$ a* network-agnostic *taxation mechanism is a mapping $\tau^{\rm na} : [0,1] \times \{\ell_e\}_{e\in\mathcal{G}} \to \{\tau_e\}$ that assigns the following flow-dependent taxation function to edge $e$:*

$$\tau_e(f_e) = \tau^{\rm na}\left(f_e; \ell_e\right) \tag{15}$$

*where $\tau^{\rm na}\left(f, \ell\right)$ satisfies the following additivity condition:*[4] *for all $e, e' \in \mathcal{G}$ and $f \in [0,1]$,*

$$\tau^{\rm na}\left(f; \ell_e + \ell_{e'}\right) = \tau^{\rm na}\left(f; \ell_e\right) + \tau^{\rm na}\left(f; \ell_{e'}\right). \tag{16}$$

Thus, both marginal-cost tolls (6) and universal tolls (7) are network-agnostic according to Definition 2.

Our goal is to derive the bounded network-agnostic taxation mechanism that minimizes the worst-case selfish routing on $\mathcal{G}^p$. We define the price of anarchy with respect to class of problems $\mathcal{G}$ and bound $T$ as the best price of anarchy we can achieve on $\mathcal{G}$ with a taxation mechanism bounded by $T$:

$$\mathrm{PoA}_T\left(\mathcal{G}\right) \triangleq \inf_{\tau \in \mathcal{T}(T,\mathcal{G})} \left\{ \sup_{G\in\mathcal{G}} \mathrm{PoA}\left(G, \tau\right) \right\}. \tag{17}$$

[3]This is essentially a regularity condition that prevents the creation of unrealistic, highly-pathological networks. For example, if a network contains an edge with a very high constant latency function, tolling functions could cause highly-sensitive users to divert to this edge, causing gross network "inefficiencies." Note that we can always assign infinite tolls to such unused edges to ensure that the regularity condition is met.

[4]The additivity condition in Definition 2 requires that two edges connected in series will be assigned the same taxation function as if they were replaced by a single edge whose latency function is the sum of the underlying latency functions. It ensures that the incentive design process be independent of network specifications, isolating the role of network information in the design process.

**Theorem 3.** *Let $\mathcal{G}(\bar{a}, \bar{b}) \subset \mathcal{G}^p$ be some subset of parallel, affine-cost networks with finite $\bar{a}$ and $\bar{b}$. For any toll bound $T$ and $S_{\rm U} \geq S_{\rm L} > 0$, define the set of universal parameters by the tuple $U_T = \left(S_{\rm L}, S_{\rm U}, \bar{a}, \bar{b}\right)$. Then there exist functions $\kappa_1\left(U_T\right)$ and $\kappa_2\left(U_T\right)$ such that the optimal network-agnostic taxation mechanism bounded by $T$ on $\mathcal{G}(\bar{a}, \bar{b})$ assigns tolling functions*

$$\tau_e(f_e) = \kappa_1(U_T) a_e f_e + \kappa_2(U_T) b_e. \tag{18}$$

*Furthermore, the price of anarchy $\mathrm{PoA}_T\left(\mathcal{G}\left(\bar{a}, \bar{b}\right)\right)$ is given by the following:*

$$\begin{cases} \frac{4}{3}\left(1 - \frac{\kappa_1(U_T)S_{\rm L}}{(1+\kappa_1(U_T)S_{\rm L})^2}\right) & \text{if } \kappa_1(U_T) < \frac{1}{\sqrt{S_{\rm L}S_{\rm U}}} \\ \frac{4}{3}\left(1 - \frac{(1+\kappa_1(U_T)S_{\rm L})\left(\frac{S_{\rm L}}{S_{\rm U}}+\kappa_1(U_T)S_{\rm L}\right)}{\left(1+2\kappa_1(U_T)S_{\rm L}+\frac{S_{\rm L}}{S_{\rm U}}\right)^2}\right) & \text{if } \kappa_1(U_T) \geq \frac{1}{\sqrt{S_{\rm L}S_{\rm U}}}. \end{cases} \tag{19}$$

See Figure 4 for a comparison of the price of anarchy afforded by Theorems 1 and 3. Note that the tolls of Theorem 3 incentivize considerably lower system costs than those of Theorem 1; this is due to the fact that Theorem 3 is optimized for a smaller class of networks.

For the reader's convenience, we include a closed-form expression for $\kappa_1(\cdot)$ in the appendix as (43), and for $\kappa_2(\cdot)$ in the proof of Theorem 3 as (27). It is evident from these expressions that $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are both nondecreasing and unbounded in $T$; among other things, this implies that $\lim_{T\to\infty} \mathrm{PoA}_T\left(\mathcal{G}\left(\bar{a}, \bar{b}\right)\right) = 1$.

We now proceed with the proof of Theorem 3, which relies on two supporting lemmas. For our first milestone, we restrict attention to simple affine taxation functions:

**Lemma 2.1.** *Let $\tau^A(\kappa_1, \kappa_2)$ denote an affine taxation mechanism that assigns tolling functions $\tau_e(f_e) = \kappa_1 a_e f_e + \kappa_2 b_e$. For any $\kappa_{\max} \geq 0$, the optimal coefficients $\kappa_1^*$ and $\kappa_2^*$ satisfying*

$$(\kappa_1^*, \kappa_2^*) \in \mathop{\arg\min}_{\kappa_1,\kappa_2 \leq \kappa_{\max}} \left\{ \sup_{G\in\mathcal{G}^p} \mathrm{PoA}\left(G, \tau^A(\kappa_1, \kappa_2)\right) \right\} \tag{20}$$

*are given by*

$$\kappa_1^* = \kappa_{\max}, \tag{21}$$

$$\kappa_2^* = \max\left\{0,\ \frac{\kappa_{\max}^2 S_{\rm L} S_{\rm U} - 1}{S_{\rm L} + S_{\rm U} + 2\kappa_{\max} S_{\rm L} S_{\rm U}}\right\}. \tag{22}$$

*Furthermore, for any $G \in \mathcal{G}^p$, $\mathrm{PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right)$ is upper-bounded by the following expression:*

$$\begin{cases} \frac{4}{3}\left(1-\frac{\kappa_{\max}S_{\rm L}}{(1+\kappa_{\max}S_{\rm L})^2}\right) & \text{if} \quad \kappa_{\max} < \frac{1}{\sqrt{S_{\rm L}S_{\rm U}}} \\ \frac{4}{3}\left(1-\frac{(1+\kappa_{\max}S_{\rm L})\left(\frac{S_{\rm L}}{S_{\rm U}}+\kappa_{\max}S_{\rm L}\right)}{\left(1+2\kappa_{\max}S_{\rm L}+\frac{S_{\rm L}}{S_{\rm U}}\right)^2}\right) & \text{if} \quad \kappa_{\max} \geq \frac{1}{\sqrt{S_{\rm L}S_{\rm U}}}. \end{cases} \tag{23}$$

See the Appendix for the proof of Lemma 2.1.

Next, in Lemma 2.2, we investigate the possibility that some other taxation mechanism could perform better than the affine $\tau^A(\kappa_1^*, \kappa_2^*)$ while still respecting the bound $T$. To that end, we assume that some arbitrary taxation mechanism outperforms affine tolls, and deduce various properties of these hypothetical tolls. We show that this hypothetical "better" taxation mechanism must universally charge higher tolls than our optimal affine tolls.

**Lemma 2.2.** *Let $\tau^*$ be any network-agnostic taxation mechanism such that for $\kappa_{\max} \geq 0$*

$$\sup_{G\in\mathcal{G}^p} \mathrm{PoA}\left(\mathcal{G}^p, \tau^*\right) < \sup_{G\in\mathcal{G}^p} \mathrm{PoA}\left(\mathcal{G}^p, \tau^A(\kappa_1^*, \kappa_2^*)\right). \tag{24}$$

*Then $\tau^*$ must charge strictly higher tolls than $\tau^A(\kappa_1^*, \kappa_2^*)$ on every edge in every network:*

$$\forall\ e \in \mathcal{G}^p, \quad \forall\ f_e \in (0,1], \quad \tau_e^*(f_e) > \tau_e^A(f_e). \tag{25}$$

The proof of Lemma 2.2 appears in the Appendix.

*Proof of Theorem 3.* For any non-negative $\kappa_1$ and $\kappa_2$, $\tau^A(\kappa_1, \kappa_2)$ is tightly bounded by $\left(\kappa_1\bar{a} + \kappa_2\bar{b}\right)$ on $\mathcal{G}\left(\bar{a}, \bar{b}\right)$. Note that for $\kappa_1^*$ and $\kappa_2^*$ as defined in Lemma 2.1, $\left(\kappa_1^*\bar{a} + \kappa_2^*\bar{b}\right)$ is a strictly increasing, continuous function of $\kappa_{\max}$. Thus, for any $T \geq 0$, there is a unique $\kappa_{\max}^* \geq 0$ for which $\tau^A(\kappa_1^*, \kappa_2^*)$ is tightly bounded by $T$ on $\mathcal{G}\left(\bar{a}, \bar{b}\right)$. We define the function $\kappa_1(U_T)$ as the maximal $\kappa_{\max}^*$ for any $T \geq 0$, given $S_{\rm L}, S_{\rm U}, \bar{a}$, and $\bar{b}$. That is, $\kappa_1(U_T)$ is defined implicitly as the unique function satisfying

$$\kappa_1(U_T)\bar{a} + \max\left\{0, \frac{\left(\kappa_1^2(U_T)S_{\rm L}S_{\rm U} - 1\right)\bar{b}}{S_{\rm L} + S_{\rm U} + 2\kappa_1(U_T)S_{\rm L}S_{\rm U}}\right\} = T. \tag{26}$$

For completeness, in the appendix we include a closed-form expression for $\kappa_1(U_T)$ as (43). We define $\kappa_2(U_T)$ as

$$\kappa_2(U_T) = \max\left\{0,\ \frac{\kappa_1^2(U_T)S_{\rm L}S_{\rm U} - 1}{S_{\rm L} + S_{\rm U} + 2\kappa_1(U_T)S_{\rm L}S_{\rm U}}\right\}. \tag{27}$$

Let $e' \in \bar{\mathcal{G}}$ be an edge with latency function $\ell_{e'}(f_{e'}) = \bar{a}f_{e'} + \bar{b}$. By construction, the tolling function assigned by $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$ to $e'$ satisfies bound $T$ with equality: $\tau_{e'}^A(1) = T$.

Now let $\tau^*$ be any taxation mechanism with a strictly lower price of anarchy than $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$. By Lemma 2.2, $\tau^*$ assigns higher tolling functions than $\tau^A(\kappa_1(U_T), \kappa_2(U_T))$ on *every* edge for every flow rate. In particular, on edge $e'$, $\tau_{e'}^*(1) > \tau_{e'}^A(1) = T$, violating bound $T$ and proving the optimality of $\tau^A\left(\kappa_1(U_T), \kappa_2(U_T)\right)$ over the space of all network-agnostic taxation mechanisms bounded by $T$. By substituting $\kappa_1(U_T)$ for $\kappa_{\max}$ in expression (23), we obtain the complete price of anarchy expression (19). □

It may be helpful to note that the crucial point in Theorem 3 is that the upper bound $T$ allows us to compute a maximum tolling coefficient $\kappa_{\max}$; it is this $\kappa_{\max}$ that enters the price of anarchy expression in (19). Thus, an alternative formulation of boundedness is possible which simply specifies a $\kappa_{\max}$ and dispenses with specifying $T$, $\bar{a}$, and $\bar{b}$. This formulation represents a relative boundedness in which tolls cannot be too much larger than realized latency function parameters.

## VI. Negative Results for Fixed Tolls

Theorem 3 showed that simple affine tolling functions are sufficient to achieve the best-possible price of anarchy for network-agnostic bounded taxation mechanisms. It is natural to ask what guarantees are possible for an even simpler class of taxation functions, the constant functions. There are practical benefits to such fixed tolls, foremost among which is the simplicity and predictability they offer to network users.

It has long been known that flow-varying tolls are sufficient to optimize network routing in cases when the network topology is unknown [22]. We ask here if fixed tolls can provide the same guarantee; i.e., we ask if (strictly) flow-varying tolls are also *necessary* to optimize routing in these settings. In Theorem 4, we prove this necessity, which immediately implies that the price of anarchy of network-agnostic fixed tolls is bounded away from 1.

**Theorem 4.** *If for every $G \in \mathcal{G}$ and unit-sensitivity homogeneous population $s$, network-agnostic taxation mechanism $\tau$ satisfies*

$$\mathcal{L}^{\rm nf}(G, s, \tau) = \mathcal{L}^*(G), \tag{28}$$

*then it must be the case that $\tau$ assigns strictly flow-varying taxation functions to some network edges.*

*Proof.* We prove Theorem 4 by contradiction. Let $\tau^{\rm na}$ be a network-agnostic fixed tolling mechanism for which $\mathcal{L}^{\rm nf}(G, s, \tau^{\rm na}) = \mathcal{L}^*(G)$; that is, it is a mapping from latency functions to non-negative constant taxation functions that enforces optimal routing on *every* network. Consider the two-path network shown in Figure 5(a); denote this network $G_n$. The upper path is composed of $n$ copies of the same link in series; network-agnosticity requires that $\tau^{\rm na}$ charges the same toll to every copy of that link. For a total traffic mass of $r$, the optimal routing profile for this network is $f_1^* = b/2$ and $f_2^* = r - b/2$. For a unit-sensitivity homogeneous population, optimal fixed tolls $\tau_1$ and $\tau_2$ must satisfy

$$\tau_2 = n\tau_1 - b/2. \tag{29}$$

Since these tolls are network-agnostic, $\tau_1$ cannot be a function of $b$, so there exists some universal constant $\beta > 0$ for which $\tau_1 = \beta$ and $\tau_2 = n\beta - b/2$. It is straightforward to show that for any $n$ and any choice of $\beta$, these tolls induce optimal routing on the network for a unit-sensitivity homogeneous user population. That is, $\mathcal{L}^{\rm nf}(G_n, s, \tau^{\rm na}) = \mathcal{L}^*(G_n)$

Our hope is that these tolling functions would optimize routing when applied to *any* network; i.e., that we could apply $\tau_1 = \beta$ to any edge with latency function $\ell_e(f_e) = f_e$, and $\tau_2$ to any edge with latency function $b$ and still get optimal performance. To test this, we apply the same tolls

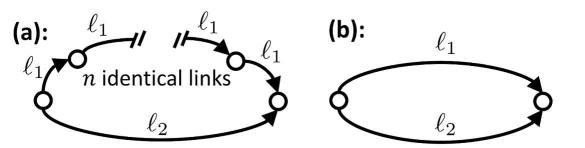


Fig. 5. Networks for Theorem 4 Proof. In both networks, $\ell_1(f_1) = f_1$ and $\ell_2(f_2) = b$, with $b > 0$. (a): Network with $n$ copies of the same link in series on the upper path. Since every upper edge has the same latency function, network-agnostic tolls must charge the same amount to each edge. (b): The same network with $n = 1$; if network-agnostic tolls $\tau_1$ and $\tau_2$ were designed for network (a), they can cause highly inefficient performance on network (b).

to the network in Figure 5(b), which we denote $G_1$. Here, we find that $\tau_2 = n\beta - b/2$ is now much too high; if the total traffic rate is high enough, these tolls induce a flow with $f_1 = \beta(n-1) + b/2$ and $f_2 = 0$, even though the optimal flow has $f_1 = b/2$. This allows us to compute a lower bound on the price of anarchy for these tolling functions:

$$\frac{\mathcal{L}^{\rm nf}(G_1, s, \tau^{\rm na})}{\mathcal{L}^*(G_1)} \geq \frac{\left(\beta(n-1) + \frac{b}{2}\right)^2}{b\left(\beta(n-1) + \frac{b}{4}\right)}, \tag{30}$$

which is unbounded in both $n$ and $\beta$, generating a contraction to our hypothesis that for all $G$, $\mathcal{L}^{\rm nf}(G, s, \tau^{\rm na}) = \mathcal{L}^*(G)$. □

In light of this negative result, in Theorem 5, we ask what guarantees are possible with fixed tolls if we know the network structure but do not know the user sensitivities; refer to the last row of Table II for a quick summary of the setting we investigate here. Since we are allowing these fixed tolls to depend on network structure (e.g., the number of edges in the network), we denote such taxation functions by $\tau^{\rm ft}(G) = \{\tau_e^{\rm ft}(G)\}_{e\in G}$. The following theorem demonstrates that any network-dependent fixed-toll taxation mechanism generally provides poor performance guarantees when compared with the optimal bounded taxation mechanism from Theorem 3.

**Theorem 5.** *Consider any network-dependent fixed-toll taxation mechanism $\tau^{\rm ft}$. For any network $G \in \mathcal{G}^p$,*

$$\sup_{s\in\mathcal{S}} \mathcal{L}^{\rm nf}\left(G, s, \tau^{\rm ft}(G)\right) \geq \sup_{s\in\mathcal{S}} \mathcal{L}^{\rm nf}\left(G, s, \tau^A(1/S_{\rm U}, 0)\right), \tag{31}$$

*with affine tolls $\tau^A(\cdot)$ as defined in Lemma 2.1. Thus,*

$$\sup_{G\in\mathcal{G}} \mathrm{PoA}\left(G, \tau^{\rm ft}\right) \geq \sup_{G\in\mathcal{G}^p} \mathrm{PoA}\left(G, \tau^A(1/S_{\rm U}, 0)\right) = \frac{4}{3}\left(1 - \frac{S_{\rm L}/S_{\rm U}}{(1 + S_{\rm L}/S_{\rm U})^2}\right). \tag{32}$$

We point out that the right-hand side of (32) represents the price of anarchy due to network-agnostic affine tolls for a *very low toll upper bound.* For example, in the canonical Pigou network depicted in Figure 4, if $S_{\rm U} = 10$, affine tolls prescribed by $\tau^A(1/S_{\rm U}, 0)$ imply a toll upper-bound of just 0.1. As shown in Figure 4, the price of anarchy for optimal affine tolls is steeply decreasing in the toll upper-bound, so a designer wishing to exploit the simplicity of fixed tolls may need to accept lower performance guarantees as a result.

Furthermore, it is important to note that Theorem 5 shows that $\tau^A$, a *network-agnostic* tolling mechanism, provides better performance guarantees (even for moderately low tolls) than $\tau^{\rm ft}$, a *network-dependent* tolling mechanism. This shows the power of Theorem 3's taxation mechanism: given *less* information, it performs *better* than any fixed-toll taxation mechanism.

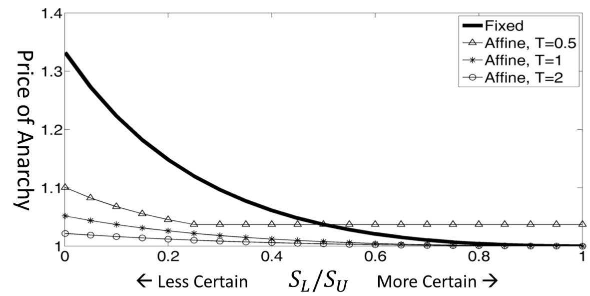


Fig. 6. Comparison of Price of Anarchy guaranteed by Theorems 3 and 5. All plots are for $S_{\rm L} = 1$ and $\bar{a} = \bar{b} = 1$. The horizontal axis represents the level of certainty in price-sensitivity; note that most taxation mechanisms guarantee a price of anarchy of 1 for complete certainty unless they are restricted by a very low upper-bound. The solid line represents the price of anarchy resulting from fixed tolls (according to (32)), and the marked lines represent the price of anarchy resulting from optimal flow-varying affine tolls for a given toll bound (according to (19)). Note that for a very low toll bound, fixed tolls slightly outperform affine tolls for well-characterized populations; this is due to the fact that the fixed tolls are not restricted by the toll upper bound.

See Figure 6 for a comparison of the price of anarchy afforded by Theorems 3 and 5, and note that fixed tolls only outperform flow-varying affine tolls when both uncertainty and the toll upper bound are low. In all other situations, optimal affine tolls provide better performance guarantees.

The proof of Theorem 5 first considers homogeneous sensitivity distributions and then extends to heterogeneous. We write $f^{\rm ft}(G, s, \tau)$ and $\mathcal{L}^{\rm nf}(G, s, \tau)$ to denote a Nash flow and its associated total latency induced by fixed tolls $\tau \in \mathbb{R}^n$ on network $G$, with homogeneous sensitivity $s \in [S_{\rm L}, S_{\rm U}]$. Similarly, we write the total latency of a Nash flow resulting from affine tolls $\tau^A(\kappa_1, \kappa_2)$ as $\mathcal{L}^{\rm nf}\left(G, s, \tau^A(\kappa_1, \kappa_2)\right)$.

Define the optimal fixed tolls $\tau^*$ as

$$\tau^* \in \mathop{\arg\min}_{\tau\in\mathbb{R}^n} \max_{s\in[S_{\rm L}, S_{\rm U}]} \mathcal{L}^{\rm nf}(G, s, \tau). \tag{33}$$

That is, $\tau^*$ is in the set of edge tolls that minimize the total latency for the worst possible user sensitivity.

In Lemma 5.1, we see that there is a curious relationship between the total latencies of Nash flows resulting from fixed tolls and those resulting from affine tolls $\tau^A(1/S_{\rm U}, 0)$. That is, the optimal fixed tolls guarantee the same worst-case performance as affine tolls with extremely low coefficients.

**Lemma 5.1.** *For any $G \in \mathcal{G}^p$, for a homogeneous population, the worst-case total latency resulting from the optimal fixed tolls $\tau^*$ is equal to the worst-case total latency resulting from $\tau^A(1/S_{\rm L}, 0)$:*

$$\max_{s\in[S_{\rm L}, S_{\rm U}]} \mathcal{L}^{\rm nf}\left(G, s, \tau^*\right) = \max_{s\in[S_{\rm L}, S_{\rm U}]} \mathcal{L}^{\rm nf}\left(G, s, \tau^A(1/S_{\rm U}, 0)\right). \tag{34}$$

The proof of Lemma 5.1 appears in the appendix.

*Proof of Theorem 5.* Since the set of homogeneous populations is a strict subset of the set of heterogeneous ones, we can only make things worse by extending from homogeneous to heterogeneous populations, so the bound in (32) must hold. The expression in (32) is obtained by substituting $1/S_{\rm U}$ in for $\kappa_{\max}$ in the first part of expression (23). □

TABLE II

| Toll Type | Information Available | | | Tolling Functions Required | | |
|---|---|---|---|---|---|---|
| | Topology | Demands | Sensitivities | Flow-Varying | Unbounded | Performance Guarantee |
| Fixed [20], [21] | ✓ | ✓ | ✓ | | | 100% |
| Marginal-Cost [22], [23] | | | ✓ | ✓† | | 100% |
| **Theorem 1:** Universal | | | | ✓ | ✓‡ | 100% |
| Characterization Results for various tolling-function constraints | | | | | | |
| **Theorem 3:** Bounded Affine g | | | | ✓ | | Good, increasing in toll upper bound |

The relationship between allowable tolls, informational dependencies and performance guarantees for several taxation methodologies. Simple fixed tolls require a precise system characterization to guarantee optimality. Flow-varying marginal-cost tolls guarantee optimality, requiring only knowledge of the (homogeneous) user-sensitivities. Theorem 1's universal tolls require none of this information, but may be arbitrarily large. Theorem 3 disallows unbounded tolls and derives the optimal information-independent bounded tolls for a sub-class of networks. Theorem 5 disallows even flow-varying tolls and shows that sensitivity-agnostic fixed tolls perform relatively poorly even if they are network-aware.
† The necessity of strictly flow-varying tolls in this setting is shown in Theorem 4.
‡ The necessity of unbounded tolls in this setting is an immediate corollary of Theorem 3.

## VII. Conclusion

In this paper we have explored several avenues for influencing social behavior when aspects of the underlying system are uncertain. Table II shows our results in context with previous results on this topic, illustrating the informational requirements and sophistication required of each taxation mechanism.

Avenues for future work include incorporating our results into recent studies on the price of anarchy for unknown or varying traffic rates [15], [16]. How would knowledge of traffic rate factor into our taxation designs? Furthermore, in practical problems, it may be that not every edge is available for taxation; this prompts the question of *which edges* are best-suited for taxes if other parameters are uncertain.

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## Appendix: Proofs of Supporting Lemmas

To prove Lemma 2.1, we analytically relate the Nash flows induced by affine tolls with coefficients $\kappa_1$ and $\kappa_2$ to the

Nash flows induced by marginal-cost tolls scaled by $\kappa_1$ for some other sensitivity distribution $s'$. We can then use known analytical techniques for scaled marginal-cost tolls to derive the optimal $\kappa_1$ and $\kappa_2$. We make use of the following theorem:

**Theorem 6** (Brown and Marden, [7]). *For any routing problem $G \in \mathcal{G}^p$ satisfying the assumptions of Theorem 3, the scaled marginal-cost taxation mechanism $\tau^{\rm smc}(\kappa)$ assigns the following tolls to any edge $e \in \mathcal{G}^p$ for $\kappa \geq 0$:*

$$\tau_e^{\rm smc}(f_e) = \kappa a_e f_e. \tag{35}$$

*The unique cost-minimizing marginal-cost toll scalar is*

$$\kappa^* = \frac{1}{\sqrt{S_{\rm L} S_{\rm U}}} = \underset{\kappa \geq 0}{\arg\min} \ \{{\rm PoA}(G, \tau^{\rm smc}(\kappa))\}. \tag{36}$$

*Finally, for any $G \in \mathcal{G}^p$, for $q = S_L/S_U$, the price of anarchy resulting from the optimal scaled marginal-cost taxation mechanism is*

$$\mathrm{PoA}\left(G, \tau^{\rm smc}(\kappa^*)\right) \leq \frac{4}{3}\left(1 - \frac{\sqrt{q}}{\left(1+\sqrt{q}\right)^2}\right). \tag{37}$$

*Proof of Lemma 2.1*

Let $G \in \mathcal{G}^p$ and $\kappa_1 \geq \kappa_2 \geq 0$.[5] For user $x \in [0,1]$ with sensitivity $s_x \in [S_{\rm L}, S_{\rm U}]$, the cost of edge $e \in G$ given flow $f$ under affine tolls is given by

$$J_x^e(f) = (1+\kappa_1 s_x) a_e f_e + (1+\kappa_2 s_x) b_e.$$

Note that we may scale $J_x^e(f)$ by any edge-independent factor without changing the underlying preferences of agent $x$. Thus, without loss of generality, we may write

$$J_x^e(f) = \frac{1+\kappa_1 s_x}{1+\kappa_2 s_x} a_e f_e + b_e. \tag{38}$$

Now, define sensitivity distribution $s'$ by the following: for any $x \in [0,1]$, $s'_x$ satisfies

$$s'_x = \frac{s_x(\kappa_1 - \kappa_2)}{\kappa_1(1+\kappa_2 s_x)}. \tag{39}$$

By a series of algebraic manipulations, we may combine (38) and (39) to obtain

$$J_x^e(f) = \left(1+\kappa_1 s'_x\right) a_e f_e + b_e, \tag{40}$$

which is simply the cost resulting from scaled marginal-cost tolls (35). Thus, for any sensitivity distribution $s$, we may model a Nash flow resulting from affine tolls with coefficients $\kappa_1$ and $\kappa_2$ as a Nash flow for sensitivity distribution $s'$ resulting from scaled marginal-cost tolls with $\kappa = \kappa_1$.

Thus, by Theorem 6, assuming first that $\kappa_{\max}$ is sufficiently high, our optimal choice of $\kappa_1$ is that which satisfies

$$\kappa_1 = \frac{1}{\sqrt{S'_{\rm L} S'_{\rm U}}}, \tag{41}$$

where $S'_{\rm L}$ and $S'_{\rm U}$ are computed according to (39).

Combining (39) and (41) yields the following characterization of the optimal $\kappa_2$ with respect to $\kappa_1$, for $\kappa_{\max} \geq (S_{\rm L}S_{\rm U})^{-1/2}$:

$$\kappa_2 = \frac{\kappa_1^2 S_{\rm L} S_{\rm U} - 1}{S_{\rm L} + S_{\rm U} + 2\kappa_1 S_{\rm L} S_{\rm U}}. \tag{42}$$

Evaluating (37) at $q = S'_{\rm L}/S'_{\rm U}$ verifies the second part of (23) as the correct expression for $\mathrm{PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right)$ when $\kappa_{\max}$ is large.

Consider the case when $\kappa_{\max} < (S_{\rm L}S_{\rm U})^{-1/2}$. Now, (42) would prescribe a negative value for $\kappa_2$, so the optimal choice is to let $\kappa_2$ saturate at 0. Now, we are precisely applying scaled marginal-cost tolls with $\kappa = \kappa_1$, so we apply the fact shown in Lemma 1.2 of [7] that on this class of networks, if $\kappa \leq (S_{\rm L}S_{\rm U})^{-1/2}$, the worst-case total latency of a Nash flow always occurs for the extreme low-sensitivity homogeneous sensitivity distribution given by $s_x \equiv S_{\rm L}$ for all $x \in [0,1]$.

Equation (35) in [7] gives the total latency of a Nash flow for a homogeneous population with sensitivity $S_{\rm L}$ as

$$\mathcal{L}^{\rm nf}(G, S_{\rm L}, \kappa) = L_R - \frac{\kappa S_{\rm L}}{(1+\kappa S_{\rm L})^2}\Theta, \tag{44}$$

where $L_R$ and $\Theta$ are positive constants depending only on $G$, satisfying $\Theta \leq L_R$. It is easy to verify that the above expression is minimized on a subset of $\left[0, (S_{\rm L}S_{\rm U})^{-1/2}\right]$ by maximizing $\kappa$, and using the fact that $\Theta \leq L_R$, we may verify that the price of anarchy for $\kappa_{\max} < (S_{\rm L}S_{\rm U})^{-1/2}$ is given by the first part of (23), completing the proof of Lemma 2.1. □

*Proof of Lemma 2.2*

Here, we derive properties of any taxation mechanism that outperforms $\tau^A(\kappa_1^*, \kappa_2^*)$. We define the set of routing problems $\mathcal{G}^0$ as follows: $G \in \mathcal{G}^0$ is a parallel network consisting of two edges, with $\ell_1(f_1) = cf_1$ and $\ell_2(f_2) = c$.

Let $G \in \mathcal{G}^0$. For any $c$, the optimal flow on $G$ is $(f_1^*, f_2^*) = (1/2, 1/2)$ and the optimal total latency is $\mathcal{L}^*(G) = 3c/4$, but the un-tolled Nash flow has a total latency of $\mathcal{L}^{\rm nf}(G, s, \emptyset) = c$, so the un-tolled price of anarchy is $4/3$. It is straightforward to show furthermore that if $S_{\rm U} > S_{\rm L} \geq 0$, for any $\kappa_{\max} > 0$, this network constitutes a worst-case example and the price of anarchy bound of this particular network is tight; i.e., it equals the expression given in (23): $\mathrm{PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right) = \sup_{G \in \mathcal{G}^p} \mathrm{PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right)$. Thus, if our hypothetical $\tau^*$ outperforms $\tau^A$ in general, it must specifically outperform $\tau^A$ on any network $G \in \mathcal{G}^0$, or

$$\mathrm{PoA}\left(G, \tau^*\right) < \mathrm{PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right). \tag{45}$$

Now, we investigate the performance of the hypothetical tolling mechanism $\tau^*$ on networks in $\mathcal{G}^0$. Given a network $G \in \mathcal{G}^0$, $\tau^*$ assigns edge tolling functions $\tau_1^*(f_1)$ and $\tau_2^*(f_2)$. Recall that since $\tau^*$ is network-agnostic, there is some function $\tau^*(f; a, b)$ such that an edge $e \in E$ with latency function $\ell_e(f_e) = a_e f_e + b_e$ is assigned tolling function $\tau^*(f_e; a_e, b_e)$. By analyzing networks in $\mathcal{G}^0$, we can deduce properties of the function with the 2nd and 3rd arguments set to 0, since $\tau_1^*(f_1) = \tau^*(f_1; c, 0)$ and $\tau_2^*(f_2) = \tau^*(f_2; 0, c)$.

Now we show that $\tau^*$ must assign higher tolls than $\tau^A(\kappa_1^*, \kappa_2^*)$. Let $S_{\rm U} > S_{\rm L}$. By design, the worst-case Nash flows resulting from $\tau^A(\kappa_1^*, \kappa_2^*)$ occur for homogeneous populations with $s = S_{\rm L}$ and $s = S_{\rm U}$. Since any network $G \in \mathcal{G}^0$ has only 2 links, we can characterize a Nash flow simply by

[5] Here, the requirement that $\kappa_1 \geq \kappa_2$ is without loss of generality; later analysis shows that $\kappa_2 > \kappa_1$ would always result in a Nash flow with higher congestion than the un-tolled case.

$$\kappa_1(U) = \min\left\{\frac{T}{\bar{a}},\ \frac{2TS_{\mathrm{L}}S_{\mathrm{U}} - (S_{\mathrm{L}}+S_{\mathrm{U}})\bar{a} + \sqrt{\left((S_{\mathrm{L}}+S_{\mathrm{U}})\,\bar{a} + 2TS_{\mathrm{L}}S_{\mathrm{U}}\right)^2 + 4\bar{b}S_{\mathrm{L}}S_{\mathrm{U}}\left(2\bar{a}+\bar{b}+T\left(S_{\mathrm{L}}+S_{\mathrm{U}}\right)\right)}}{2S_{\mathrm{L}}S_{\mathrm{U}}\left(2\bar{a}+\bar{b}\right)}\right\} \quad (43)$$

Fig. 7. Closed-form expression for $\kappa_1(U)$ used in Theorem 3. Note that it is a continuous, unbounded, strictly increasing function of $T$.

the flow on edge 1; accordingly, let $f_{\mathrm{L}}(c)$ denote the flow as a function of $c$ on edge 1 in the Nash flow resulting from sensitivity distribution $s = S_{\mathrm{L}}$, and $f_{\mathrm{H}}(c)$ the corresponding edge 1 flow for $s = S_{\mathrm{U}}$. These flows are the solutions to the following equations:

$$cf_{\mathrm{L}}(c)\left(1+\kappa_1^* S_{\mathrm{L}}\right) = c\left(1+\kappa_2^* S_{\mathrm{L}}\right), \quad (46)$$
$$cf_{\mathrm{H}}(c)\left(1+\kappa_1^* S_{\mathrm{U}}\right) = c\left(1+\kappa_2^* S_{\mathrm{U}}\right). \quad (47)$$

Summing (46) and (47) yields

$$\kappa_1^*\left(f_{\mathrm{L}}(c) - f_{\mathrm{H}}(c)\right) = \frac{f_{\mathrm{H}}(c)}{S_{\mathrm{U}}} - \frac{f_{\mathrm{L}}(c)}{S_{\mathrm{L}}} + \frac{1}{S_{\mathrm{L}}} - \frac{1}{S_{\mathrm{U}}}. \quad (48)$$

It is always true that $f_{\mathrm{H}}(c) < f_{\mathrm{L}}(c)$. By design, $\mathcal{L}(f_{\mathrm{L}}(c)) = \mathcal{L}(f_{\mathrm{H}}(c))$. Note that $\mathcal{L}$ is simply a concave-up parabola in the flow on edge 1.

Now, let $f_{\mathrm{L}}^*(c)$ and $f_{\mathrm{H}}^*(c)$ be defined as the Nash flows resulting from $\tau^*$ for a given value of $c$; i.e., the solutions to

$$cf_{\mathrm{L}}^*(c) + \tau_1^*(f_{\mathrm{L}}^*(c))S_{\mathrm{L}} = c + \tau_2^*(1-f_{\mathrm{L}}^*(c))S_{\mathrm{L}}, \quad (49)$$
$$cf_{\mathrm{H}}^*(c) + \tau_1^*(f_{\mathrm{H}}^*(c))S_{\mathrm{U}} = c + \tau_2^*(1-f_{\mathrm{H}}^*(c))S_{\mathrm{U}}. \quad (50)$$

Since $\tau^*$ guarantees better performance than $\tau^A(\kappa_1^*, \kappa_2^*)$, it must do so in particular for these homogeneous sensitivity distributions $s = S_{\mathrm{L}}$ and $s = S_{\mathrm{U}}$. Since $\mathcal{L}$ is a parabola, this means that for any $c$, $f_{\mathrm{H}}(c) < f_{\mathrm{H}}^*(c) < f_{\mathrm{L}}^*(c) < f_{\mathrm{L}}(c)$.

Define the nondecreasing function $\Delta^*(f) = \tau_2^*(f) - \tau_1^*(1-f)$ (which is implicitly also a function of $c$), so equations (49) and (50) can be combined and rearranged to show

$$\Delta^*(f_{\mathrm{L}}^*(c)) - \Delta^*(f_{\mathrm{H}}^*(c)) > c\left[\frac{f_{\mathrm{H}}(c)}{S_{\mathrm{U}}} - \frac{f_{\mathrm{L}}(c)}{S_{\mathrm{L}}} + \frac{1}{S_{\mathrm{L}}} - \frac{1}{S_{\mathrm{U}}}\right]$$
$$= \kappa_1^* c\left(f_{\mathrm{L}}(c) - f_{\mathrm{H}}(c)\right) \quad (51)$$

The above inequality can be further loosened by replacing $f_{\mathrm{L}}^*(c)$ with $f_{\mathrm{L}}(c)$ and $f_{\mathrm{H}}^*(c)$ with $f_{\mathrm{H}}(c)$, and substituting from (48) and rearranging, we finally obtain

$$\frac{\Delta^*(f_{\mathrm{L}}(c)) - \Delta^*(f_{\mathrm{H}}(c))}{f_{\mathrm{L}}(c) - f_{\mathrm{H}}(c)} > \kappa_1^* c. \quad (52)$$

Since this must be true for any $c > 0$, the average slope of $\Delta^*(f)$ must be greater than $\kappa_1^* c$ for all $f > 0$. Since $\tau_2^*(f) \geq 0$ this implies that $\tau_1^*(f) > \kappa_1^* cf$ for all $f > 0$, or that

$$\tau^*(f; a, 0) > \tau^A(f; a, 0) \quad (53)$$

for all positive $f$ and $a$.

Now consider the following rearrangement of (50):

$$\tau_2^*(1-f_{\mathrm{H}}^*(c)) = \left[cf_{\mathrm{H}}^*(c) + \tau_1^*(f_{\mathrm{H}}^*(c)) - cS_{\mathrm{U}}\right]\cdot\frac{1}{S_{\mathrm{U}}}$$
$$> c\left[\left(1+\kappa_1^* S_{\mathrm{U}}\right)f_{\mathrm{H}}(c) - 1\right]\cdot\frac{1}{S_{\mathrm{U}}}$$
$$= \kappa_2^* c S_{\mathrm{U}} = \tau_2^A(f). \quad (54)$$

This implies that $\tau_2^*(f) > \kappa_2^* c$ for all $f > 0$, or that

$$\tau^*(f; 0, b) > \tau^A(f; 0, b) \quad (55)$$

for all positive $f$ and $b$.

Finally, note that the additivity assumption of Definition 2 implies that $\tau^*(f; a, b)$ is additive in its second and third arguments. That is, we may add inequalities (53) and (55) to conclude that for all nonnegative $f$, $a$, and $b$, it is true that

$$\tau^*(f; a, b) > \kappa_1^* af + \kappa_2^* b, \quad (56)$$

or that a necessary condition for $\sup_{G\in\mathcal{G}^p}\mathrm{PoA}(G, \tau^*) < \sup_{G\in\mathcal{G}^p}\mathrm{PoA}(G, \tau^A)$ is that $\tau^*$ must charge higher tolls on *every* edge in *every* network. $\square$

*Proof of Lemma 5.1*

We first derive a simple expression for a Nash flow for a homogeneous population as a linear function of the tolls $\tau$. Note that in the context of fixed tolls, Nash flows are unique in cost: for a given routing game, every Nash flow exhibits the same cost on all edges [19].

**Claim 5.1.1.** *A Nash flow on $G \in \mathcal{G}$ for sensitivity $s \in \mathcal{S}_1$ and fixed tolls $\tau \in \mathbb{R}^n$ that has positive traffic on all links can be described by the following linear function:*

$$f^{\mathrm{ft}}(G, s, \tau) = R + H(b + s\tau), \quad (57)$$

*where $R \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n\times n}$ are constant matrices depending only on $G$. The total latency of this flow is given by the following convex quadratic in $\tau$:*

$$\mathcal{L}^{\mathrm{ft}}(G, s, \tau) = L_R + s\tau^T H^T(2AH + I)b + s^2\tau^T H^T AH\tau. \quad (58)$$

*Proof.* Since all users share the same sensitivity, all links have equal cost to all agents in a Nash flow, so when all network edges have positive flow, for any $e_i, e_j \in E$,

$$a_i f_i + b_i + s\tau_i = a_j f_j + b_j + s\tau_j.$$

Similar to the approach in the proof of Lemma 1.2 in [7], a Nash flow $f^{\mathrm{ft}}(G, s, \tau)$ is a solution $f$ to the linear system

$$\underbrace{\begin{bmatrix} a_1 & -a_2 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{P} f = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{r} + \underbrace{\begin{bmatrix} -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{X} (b + s\tau). \quad (59)$$

$P$ is invertible, so letting $H = P^{-1}X$ and $R = P^{-1}r$, a Nash flow is given by the linear equation (57).

The following observations will be helpful to our proof:

**Observation 5.1.** *The matrices $H$ and $R$ possess the following properties for any $G \in \mathcal{G}$:*

$$\mathbf{1}^{\mathrm{T}} H b = \mathbf{0}^{\mathrm{T}}, \tag{60}$$

$$\mathbf{1}^{\mathrm{T}} R = 1, \tag{61}$$

$$AR \in \mathrm{sp}\left\{\mathbf{1}\right\}, \tag{62}$$

$$b^T H^T A H b = -M^T b. \tag{63}$$

*Finally, every column of $(AH + I)$ is in* $\mathrm{sp}\{\mathbf{1}\}$.

These facts follow algebraically from the fact that by definition, $f^{\mathrm{ft}}(G, s, \tau)$ satisfies (59). Subsutiting (57) into (1) and simplifying using the facts in Observation 5.1 yields (58). □

Next, we establish a necessary condition for a set of fixed tolls to be optimal in the sense of (33).

**Claim 5.1.2.** *Fixed tolls $\tau^*$ satisfying* (33) *must also satisfy*

$$H\left(\tau^* + \frac{b}{S_{\mathrm{L}} + S_{\mathrm{U}}}\right) = \mathbf{0}. \tag{64}$$

*Proof.* By (58) the total latency due to fixed tolls is a concave parabola in $s$, so for any $\tau$, the maximum total latency on $[S_{\mathrm{L}}, S_{\mathrm{U}}]$ occurs at either $S_{\mathrm{L}}$ or $S_{\mathrm{U}}$. Since $\mathcal{L}^{\mathrm{ft}}(G, s, \tau)$ is continuous and convex in $\tau$, this means that $\tau^*$ must satisfy

$$\mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) = \mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{U}}, \tau^*). \tag{65}$$

Thus, for any optimal fixed tolls $\tau^*$, $\mathcal{L}^{\mathrm{ft}}(G, s, \tau^*)$ is a parabola centered at $s = \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}$:

$$\operatorname*{argmin}_{s \in [S_{\mathrm{L}}, S_{\mathrm{U}}]} \mathcal{L}^{\mathrm{ft}}(G, s, \tau^*) = (S_{\mathrm{L}} + S_{\mathrm{U}})/2. \tag{66}$$

Our goal is to find the parabola with minimum as in (66) which minimizes the values in (65).

Equation (58) implies that for all $\tau, \tau' \in \mathbb{R}^n$, $\mathcal{L}^{\mathrm{ft}}(G, 0, \tau) = \mathcal{L}^{\mathrm{ft}}(G, 0, \tau')$; that is, the $s = 0$ endpoint of the parabola has the same value for all tolls. Thus, for $\tau$ satisfying (66), $\mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) < \mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau)$ if and only if $\mathcal{L}^{\mathrm{ft}}\left(G, \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}, \tau^*\right) < \mathcal{L}^{\mathrm{ft}}\left(G, \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}, \tau\right)$.

By concavity, any tolls which result in globally optimal routing for $s = \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}$ will also be optimal in the sense of (33). It is easily verified that for a known homogeneous sensitivity $s$, any tolls $\tau$ which satisfy

$$H\left(\tau + b/(2s)\right) = \mathbf{0} \tag{67}$$

result in globally optimal routing. The proof of this is obtained by substituting (67) into the gradient (with respect to $\tau$) of $\mathcal{L}^{\mathrm{ft}}(G, s, \tau)$ and applying the facts from Observation 5.1.

Therefore, any $\tau$ which satisfies (67) with $s = \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}$ will be uncertainty-optimal. That is, $\tau^*$ satisfies (64). □

Evaluating (57) with tolls satisfying (64) yields an expression for a Nash flow induced by $\tau^*$ as a function of $s$:

$$f^{\mathrm{ft}}(G, s, \tau^*) = R + Hb\left(S_{\mathrm{L}} + S_{\mathrm{U}} - s\right) / \left(S_{\mathrm{L}} + S_{\mathrm{U}}\right), \tag{68}$$

implying that $(R + Hb)$ specifies an un-tolled Nash flow. For parallel networks, it is easy to show that every element of $R$ is non-negative; thus, since $\alpha \triangleq \left(\frac{S_{\mathrm{L}} + S_{\mathrm{U}} - s}{S_{\mathrm{L}} + S_{\mathrm{U}}}\right) \in [0, 1]$, it must be that $(R + Hb\alpha)$ represents a feasible flow.

There are two possible worst-case flows using fixed toll $\tau^*$: one when the sensitivity is $S_{\mathrm{U}}$, the other when the sensitivity is $S_{\mathrm{L}}$. In terms of (68), we write these flows as:

$$f_-^{\mathrm{ft}} = f^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) = R + Hb\left(S_{\mathrm{U}}/(S_{\mathrm{L}} + S_{\mathrm{U}})\right). \tag{69}$$

$$f_+^{\mathrm{ft}} = f^{\mathrm{ft}}(G, S_{\mathrm{U}}, \tau^*) = R + Hb\left(S_{\mathrm{L}}/(S_{\mathrm{L}} + S_{\mathrm{U}})\right). \tag{70}$$

Next we show that $f_-^{\mathrm{ft}}$ and $f_+^{\mathrm{ft}}$, the worst-case flows for optimal fixed tolls, are actually exactly equal to worst-case flows achievable with *scaled marginal-cost* tolls (35) with a particular scalar. The machinery of Claim 5.1.1 describes the Nash flows $f^{\mathrm{smc}}(G, s, \kappa)$ resulting from homogeneous sensitivity $s$ and marginal-cost tolls scaled by $\kappa > 0$:

$$f^{\mathrm{smc}}(G, s, \kappa) = R + Hb/\left(1 + s\kappa\right). \tag{71}$$

The derivation of this is straightforward; it is detailed in [7].

The *worst* worst-case flows occur when the sensitivity of the population has been grossly over- or under-estimated; for example, if a population with sensitivity $S_{\mathrm{U}}$ is using a network with $\kappa = 1/S_{\mathrm{L}}$ (and vice-versa). There are two such flows:

$$f_-^{\mathrm{smc}} = R + \frac{Hb}{1 + S_{\mathrm{L}}/S_{\mathrm{U}}} \quad \text{and} \quad f_+^{\mathrm{smc}} = R + \frac{Hb}{1 + S_{\mathrm{U}}/S_{\mathrm{L}}}.$$

Comparing the above to (69) and (70), we see that $f_-^{\mathrm{smc}} = f_-^{\mathrm{ft}}$ and $f_+^{\mathrm{smc}} = f_+^{\mathrm{ft}}$. Thus, since

$$f^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) = f^{\mathrm{smc}}(G, S_{\mathrm{L}}, 1/S_{\mathrm{U}}),$$
$$f^{\mathrm{ft}}(G, S_{\mathrm{U}}, \tau^*) = f^{\mathrm{smc}}(G, S_{\mathrm{U}}, 1/S_{\mathrm{L}}),$$

it must be true that (re-writing now in terms of affine tolls)

$$\mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{L}}, \tau^*) = \mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{L}}, \tau^A(1/S_{\mathrm{U}}, 0)), \tag{72}$$

$$\mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{U}}, \tau^*) = \mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{U}}, \tau^A(1/S_{\mathrm{L}}, 0)). \tag{73}$$

By design, (72) equals (73), so we have that

$$\max_{s \in [S_{\mathrm{L}}, S_{\mathrm{U}}]} \mathcal{L}^{\mathrm{nf}}\left(G, s, \tau^A(1/S_{\mathrm{U}}, 0)\right) = \max_{s \in [S_{\mathrm{L}}, S_{\mathrm{U}}]} \mathcal{L}^{\mathrm{nf}}\left(G, s, \tau^*\right).$$

□

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